

# Limits of the energy-momentum tensor in general relativity

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## Abstract

A limiting diagram for the Segre classification of the energy-momentum tensor is obtained and discussed in connection with a Penrose specialization diagram for the Segre types. A generalization of the coordinate-free approach to limits of Paiva *et al.* to include non-vacuum space-times is made. Geroch's work on limits of space-times is also extended. The same argument also justifies part of the procedure for classification of a given spacetime using Cartan scalars.

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## 1 Introduction

The matter content in general relativity theory is described by a second order symmetric tensor, the energy-momentum tensor. Under limiting processes, one would like to know which energy-momentum tensors might arise. A step in this study is the investigation of the limits of classes of energy-momentum tensors.

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A classification of this tensor is known according to its Segre type. It seems, therefore, important to investigate the relations among the Segre types under limiting processes.

In 1969, Geroch [6] studied limits of space-times (see also [26]). Among other features, he showed that the Penrose [18] specialization diagram for the Petrov classification (figure 1) is in fact a limiting diagram, in the sense that under limiting processes only space-times with the same Petrov type or one of its specializations can be reached. Recently, a coordinate-free technique for studying the limits of vacuum space-times was developed and the limits of some well known vacuum solutions were investigated [17]. In this approach the Geroch limiting diagram for the Petrov classification was extensively used. Afterwards limits of non-vacuum space-times were studied [16] and the Penrose specialization diagram for the Segre classification [19, 20, 9, 24] was used.

The main aim of this paper is to build a limiting diagram for the Segre classification. We also compare our diagram with the Penrose specialization diagram for the Segre types, obtained through a different approach in another context. Moreover this work extends Geroch's results on limits of space-times, and it also generalizes the Paiva *et al.* [17] coordinate-free approach to limits so as to include non-vacuum space-times.

We shall use in this work the concept of limit of a space-time introduced in reference [17], wherein by a limit of a space-time, broadly speaking, we mean a limit of a family of space-times as some free parameters are taken to a limit. For instance, in the one-parameter family of Schwarzschild solutions each member is a Schwarzschild space-time with a specific value for the mass parameter  $m$ . By space-time we understand a real 4-dimensional differential manifold with a metric of signature  $(+ - - -)$  together with the attendant structures usually required in

general relativity theory [12].

We also note that although our main aim is to consider families of space-times, the same arguments about possible limits apply to the Segre type along curves within a given spacetime, and thus enable us to show that the use of these types in the classification by Cartan scalars satisfies the local constancy requirement stated by Ehlers[5]. Correspondingly, when we speak of the Segre type of a spacetime we mean the Segre type at a generic point — the limits of behaviours at special points must also be limits obtainable from the generic type.

Our major aim in the next section is to present a brief summary of Geroch's hereditary properties and discuss some basic properties of Segre classification required for section 3. In section 3 we study new hereditary properties and build a limiting diagram for the Segre classification. In section 4 we discuss our results and their applications.

## 2 Prerequisites

Geroch [6] shows that there are some properties that are inherited by all limits of a family of space-times. These properties he called hereditary. The first hereditary property devised by Geroch can be stated as follows:

**Hereditary property (Geroch):** (2.1)

Let  $T$  be a tensor or scalar field built from the metric and its derivatives. If  $T$  is zero for all members of a family of space-times, it is zero for all limits of this family.

From this property we easily conclude that the vanishing of either the Weyl or Ricci tensor or the curvature scalar are also hereditary properties. What can be said about the Petrov and Segre classifications of those tensors under limiting

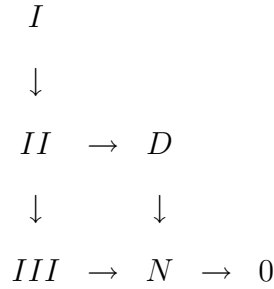


Figure 1: Limiting diagram for the Petrov classification.

processes ?

As far as the Petrov classification is concerned, using the above hereditary property, Geroch showed that although the Petrov type is not a hereditary property, “to be at least as specialized as type ...” is. In other words, the Penrose specialization diagram for the Petrov classification (figure 1) was shown to be a limiting diagram. For the sake of simplicity, in the limiting diagrams in this paper, we do not draw arrows between types whenever a compound limit exists. Thus, in figure 1, e.g., the limits  $I \rightarrow II \rightarrow D$  imply that the limit  $I \rightarrow D$  is allowed.

The Segre classification in general relativity arises from the eigenvalue problem  $(S_b^a - \lambda \delta_b^a)V^b = 0$  constructed with the trace-free Ricci tensor  $S_b^a \stackrel{def}{=} R_b^a - \frac{1}{4}\delta_b^a R$ . By virtue of Einstein’s equations,  $S_b^a$  and the energy-momentum tensor have the same Segre type. For an account of the Segre notation and Jordan matrices used throughout this article see [8, 25]. The system above has non-trivial solution only for the values of  $\lambda$  for which the characteristic polynomial

$$\det(S_b^a - \lambda \delta_b^a) \tag{2.2}$$

is equal to zero. The fundamental theorem of algebra [2] ensures that, over the

complex field, it can always be factorized as

$$(\lambda - \lambda_1)^{d_1} (\lambda - \lambda_2)^{d_2} \cdots (\lambda - \lambda_r)^{d_r}, \quad (2.3)$$

where  $\lambda_i$  ( $i = 1, 2, \dots, r$ ) are the distinct roots of the polynomial (eigenvalues), and  $d_i$  the corresponding degeneracies. To indicate the characteristic polynomial we shall introduce a list  $\{d_1 d_2 \cdots d_r\}$  of eigenvalues' degeneracies. However, when complex eigenvalues exist, instead of a digit to denote the degeneracies we shall use a letter. So, in table 1, for example, we have  $\{z\bar{z}11\}$  and  $\{2z\bar{z}\}$  instead of  $\{1111\}$  and  $\{211\}$ , respectively.

We shall now discuss the minimal polynomial, which will be important in the derivation of a limiting diagram for the Segre classification. Let  $P$  be a monic matrix polynomial of degree  $n$  in  $S_b^a$ , i.e.,

$$P = S^n + c_{n-1} S^{n-1} + c_{n-2} S^{n-2} + \cdots + c_1 S + c_0 \delta, \quad (2.4)$$

where  $\delta$  is the identity matrix and  $c_n$  are complex numbers. The polynomial  $P$  is said to be the *minimal polynomial* of  $S$  if it is the polynomial of lowest degree in  $S$  such that  $P = 0$ . It can be shown [28] that the minimal (monic) polynomial is unique and can be factorized as

$$(S - \lambda_1 \delta)^{m_1} (S - \lambda_2 \delta)^{m_2} \cdots (S - \lambda_r \delta)^{m_r}, \quad (2.5)$$

where  $m_i$  is the dimension of the Jordan submatrix of *highest dimension* for each eigenvalue  $\lambda_i$ . We shall denote the minimal polynomial in a compact form through a list  $\|m_1 m_2 \cdots m_r\|$ .

We can now find the characteristic and minimal polynomials for each Segre type. The power of the term corresponding to each eigenvalue in the characteristic polynomial is the sum of the dimensions of the Jordan submatrices with the same eigenvalue, whereas in the minimal polynomial the power is the dimension

CP $\rightarrow$	$\{1111\}$	$\{z\bar{z}11\}$	$\{211\}$	$\{2z\bar{z}\}$	$\{31\}$	$\{22\}$	$\{4\}$
MP $\downarrow$							
$\ 1111\ $	$[1,111]$	$[z\bar{z}11]$					
$\ 211\ $			$[211]$				
$\ 31\ $					$[31]$		
$\ 111\ $			$[1,1(11)]$ $[(1,1)11]$	$[z\bar{z}(11)]$			
$\ 21\ $					$[(21)1]$	$[2(11)]$	
$\ 3\ $							$[(31)]$
$\ 11\ $					$[(1,11)1]$ $[1,(111)]$	$[(1,1)(11)]$	
$\ 2\ $							$[(211)]$
$\ 1\ $							$[(1,111)]$

Table 1: Characteristic and minimal polynomials (columns - CP and rows - MP) corresponding to the Segre types in general relativity.

of the Jordan submatrix of highest dimension with that eigenvalue. Thus, for example, while Segre types  $[(1,11)1]$ ,  $[(21)1]$  and  $[31]$  have the same characteristic polynomial, denoted by  $\{31\}$ , their corresponding minimal polynomials are, respectively, given by  $\|11\|$ ,  $\|21\|$  and  $\|31\|$ . On the other hand, while the Segre types  $[2(11)]$  and  $[(21)1]$  have the same minimal polynomial, denoted by  $\|21\|$ , the associated characteristic polynomials are, respectively, given by  $\{22\}$  and  $\{31\}$ . Table 1 shows the characteristic and minimal polynomials corresponding to each Segre type.

Before closing this section we notice that: (i) the explicit expressions for the

minimal polynomial of a traceless real symmetric tensor defined on a Lorentzian space can be found in Bona *at al.* [1] and (ii) the minimal polynomial actually yields a contracted identity for  $S_b^a$ , which allows the setting up of a generalized algebraic Rainich condition for this tensor [7].

### 3 Limiting Diagram for Segre Types

In this section, we shall first discuss limiting diagrams for both the characteristic and minimal polynomials, and combine them to determine a limiting diagram for the Segre classification. Afterwards, we shall discuss new hereditary properties to refine this first version of the limiting diagram for the Segre classification.

From its definition (2.2), the characteristic polynomial is built from the metric and its derivatives. Therefore the eigenvalues, i.e., the roots of the characteristic polynomial, are scalars built from the metric  $g$  and its derivatives. The minimal polynomial, in its factorized form (2.5), is a function only of the trace-free Ricci tensor and the eigenvalues, thus it is also built from the metric and its derivatives. Since the characteristic and the minimal polynomials are, respectively, a scalar and a tensor, the hereditary property (2.1) can be applied to both.

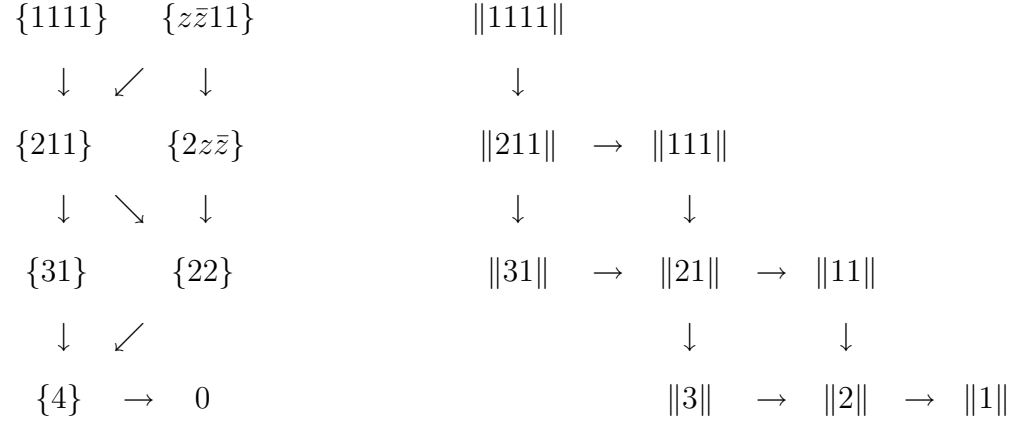
The difference between two roots of the characteristic polynomial is also a scalar which can be built with  $g$  and its derivatives. Therefore, at each degeneracy one scalar becomes zero and thus, by the hereditary property (2.1), under a limiting process, the degeneracy of the characteristic polynomial either increases or remains the same. Besides, the real and imaginary parts of complex roots are also scalars which can be built with  $g$  and its derivatives. Therefore, Segre types with real roots cannot have as a limit a Segre type with a non-real root. Further, it can be shown (see appendix A) that the limits  $\{z\bar{z}11\} \rightarrow \{1111\}$ ,  $\{2z\bar{z}\} \rightarrow \{211\}$  and  $\{2z\bar{z}\} \rightarrow \{31\}$  are forbidden. These results are summarized in the

limiting diagram for the characteristic polynomial shown in figure 2.

Let  $P_n$  be the minimal polynomial (of degree  $n$ ) of the trace-free Ricci tensor  $S_b^a$  of a family of space-times. We recall that by virtue of Einstein's equations  $S_b^a$  and the energy-momentum tensor  $T_b^a$  have the same Segre type. By definition  $P_n = 0$  for all members of this family. Thus, according to the hereditary property (2.1),  $P_n = 0$  for all limits of this family. Since the minimal polynomial is uniquely defined, for all limits of this family the minimal polynomial of the trace-free Ricci tensor  $S_b^a$  is either  $P_n$  or a lower degree polynomial. In other words, under limiting processes of a family of space-times the degree of the minimal polynomial either decreases or remains the same. Besides, from the limiting diagram for the characteristic polynomial in figure 2 we notice that the number of distinct eigenvalues either decreases or remains the same under limiting processes. Taking into account these two properties, we can work out the limiting diagram for the minimal polynomial also shown in figure 2, where the columns correspond to the same degree of the minimal polynomial, and the rows correspond to the same number of eigenvalues.

From the limiting diagrams for the characteristic and minimal polynomials in figure 2, and table 1 (which relates these polynomials to the Segre types) we can draw the first limiting diagram for the Segre classification, shown in figure 3. Indeed, starting from the limiting diagram for the minimal polynomial in figure 2, we substitute for each minimal polynomial the corresponding Segre types taken from table 1. At this point we do not take into account the character of the eigenvectors. Therefore, we represent the Segre types  $[(1,1)11]$  and  $[1,1(11)]$  by  $[(11)11]$  and the Segre types  $[(1,11)1]$  and  $[1,(111)]$  by  $[(111)1]$ . Besides, we notice that in four situations more than one Segre type is associated with the same minimal polynomial, namely to the minimal polynomial  $\|21\|$  corresponds Segre





CHARACTERISTIC POLYNOMIAL

MINIMAL POLYNOMIAL

Figure 2: Limiting diagrams for the characteristic and minimal polynomials. The minimal polynomials  $\|22\|$  and  $\|4\|$  are not shown since they cannot correspond to Segre types in general relativity [8, 12].

types  $[(21)1]$  and  $[2(11)]$ , to the minimal polynomial  $\|11\|$  corresponds Segre types  $[(111)1]$  and  $[(11)(11)]$ , to  $\|111\|$  corresponds  $[11(11)]$  and  $[z\bar{z}(11)]$ , and finally to the minimal polynomial  $\|1111\|$  corresponds the Segre types  $[1111]$  and  $[z\bar{z}11]$ .

In order to distinguish the Segre type  $[(21)1]$  from the type  $[2(11)]$ , and the Segre type  $[(111)1]$  from  $[(11)(11)]$ , we shall now consider the information in the limiting diagram of the characteristic polynomial (figure 2). We notice that while the Segre types  $[(11)11]$  may have as limit types  $[2(11)]$  and  $[(21)1]$ , the Segre type  $[31]$  may have as limit the type  $[(21)1]$  but not the type  $[2(11)]$ . Similarly, the Segre type  $[2(11)]$  may have as its limits the types  $[(31)]$  and  $[(11)(11)]$ , while the Segre type  $[(21)1]$  may have the types  $[(31)]$  and  $[(111)1]$  as its limits. This completes the limiting diagram for the Segre classification shown in figure 3 (see also ref. [15]).

It remains to study the consequences of distinguishing Segre types which differ

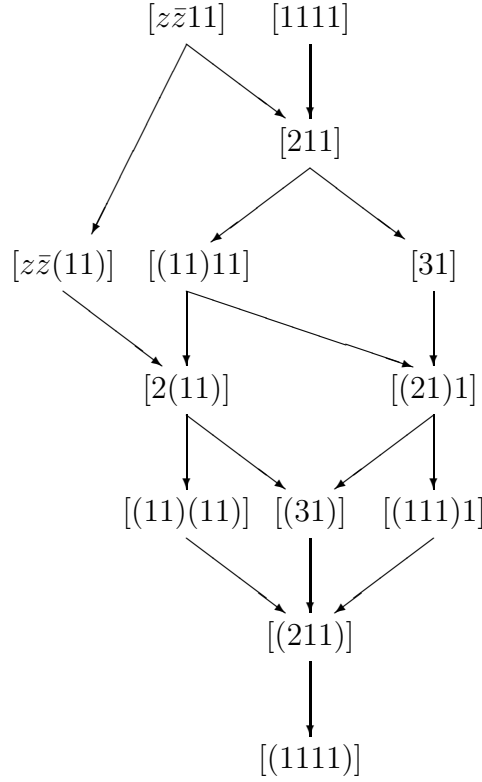


Figure 3: First diagram for the limits of Segre types. This diagram does not take into account the character of the eigenvectors.

by the character of the eigenvector. To this end, we shall now consider the type  $[(11)11]$  as representing a set of two types, namely  $[(1,1)11]$  and  $[1,1(11)]$ . Furthermore, the type  $[(111)1]$  will be looked upon as a set of the types  $[(1,11)1]$  and  $[1,(111)]$ . Firstly we will check whether type  $[(1,1)11]$  can have  $[1,1(11)]$  as its limit and vice-versa, and similarly whether the type  $[(1,11)1]$  can have the type  $[1,(111)]$  as its limit and reciprocally. Secondly, we shall find out whether the Segre types which can have as limit one of these two Segre set-types can have as limit both members of the set. Finally, we examine whether the Segre types which can be a limit of one of these two set-types can be a limit of each member of the corresponding Segre set-type.

To deal with the questions we have raised in the previous paragraph we shall introduce three hereditary properties. The first one is a corollary of the property (2.1) and can be stated as follows.

**Hereditary property:** (3.1)

Let  $E$  be a scalar field built from the metric and its derivatives. If  $E$  is nonzero and  $E > 0$  (or  $E < 0$ ) for all members of a family of space-times, then  $E \geq 0$  (or  $E \leq 0$ ) for all limits of this family.

The second hereditary property (for a proof see appendix B) we shall need can be stated as follows

**Hereditary property:** (3.2)

If the trace-free Ricci tensor has a null eigenvector for all members of a family of space-times, then all limits of this family will also have a null eigenvector.

The Segre type  $[1, (111)]$  cannot be obtained as a limit of the type  $[(1,11)1]$  because the latter has null eigenvectors and the former not. The Segre type  $[(1,11)1]$  cannot be obtained as a limit of the type  $[1, (111)]$  because this would involve a change in sign of the invariant in the Ludwig-Scanlan classification (types  $A_1$  and  $A_2$ , cf. [13, 4]). We note that this invariant also appears in Seixas [27] in a different context.

Again, a consideration of the invariants in the Ludwig-Scanlan classification types  $B$  and  $C$  together with the remark concerning the associated quartic curve classification of Penrose [19, 4] that a curve of Penrose type  $B$  cannot have as its limit a Penrose curve of type  $CC$  and vice-versa, shows that the limits  $[(1,1)11] \rightarrow [1,1(11)]$ ,  $[1,1(11)] \rightarrow [(1,1)11]$  and  $[(1,1)11] \rightarrow [2(11)]$  are forbidden.

In the diagram of figure 3 the Segre type  $[211]$  could have as a limit the generic set-type  $[(11)11]$ . Nevertheless, from the above property (3.2) we verify that  $[211]$  can have the type  $[(1,1)11]$  as limit, but it cannot have as limit the type  $[1,1(11)]$ . Similarly, in figure 3 the Segre types  $[(21)1]$ ,  $[31]$ ,  $[(1,1)11]$  and  $[211]$  could have as limit the generic set-type  $[1(111)]$ . However, the above hereditary property shows that they can have as limit the type  $[(1,11)1]$  but not the type  $[1,(111)]$ .

The last hereditary property (see appendix B for a proof) we shall need is

**Hereditary property:** (3.3)

If the trace-free Ricci tensor has a pair of complex conjugate eigenvalues for all members of a family of space-times, then all limits of this family will have either a pair of complex conjugate eigenvalues or at least a null eigenvector.

In the diagram of figure 3 the Segre type  $[z\bar{z}11]$  could have as limit the generic set-types  $[(11)11]$  and  $[1(111)]$ . However, according to the hereditary properties (3.2) and (3.3) we conclude that although the type  $[z\bar{z}11]$  can still have as its limit the types  $[(1,1)11]$  and  $[(1,11)1]$ , it cannot have as limit the types  $[1,1(11)]$  and  $[1,(111)]$ .

Table 2 summarizes the limits which were forbidden by the corresponding hereditary properties we have studied in this section. This table together with the diagram in figure 3 lead to the limiting diagram in figure 4.

## 4 Conclusion

We have built a limiting diagram for the Segre classification (figure 4) based upon hereditary properties. To achieve this goal we have extended Geroch's hereditary properties [6]. As a matter of fact, we have introduced the properties (3.1), (3.2) and (3.3), worked out limiting diagrams for the characteristic and minimal

$[1,(111)]$	$\leftrightarrow$	$[(1,11)1]$
$[1,1(11)]$	$\leftrightarrow$	$[(1,1)11]$
$[(1,1)11]$	$\rightarrow$	$[2(11)]$
$[211]$	$\rightarrow$	$[1,1(11)]$
$[(21)1]$	$\rightarrow$	$[1,(111)]$
$[31]$	$\rightarrow$	$[1,(111)]$
$[(1,1)11]$	$\rightarrow$	$[1,(111)]$
$[211]$	$\rightarrow$	$[1,(111)]$
$[z\bar{z}11]$	$\rightarrow$	$[1,(111)]$
$[z\bar{z}11]$	$\rightarrow$	$[1,1(11)]$

Table 2: Limits forbidden by the hereditary properties 3.1, 3.2 and 3.3.

polynomials (figure 2), and used them to construct the limiting diagram shown in figure 4.

The *limiting* diagram for the Segre types we have studied in this article essentially coincides with the Penrose *specialization* diagram for the Segre classification [19, 9]. As the Penrose specialization is an inverse relation to deformation [24], the Sánchez-Plebański-Przanowski deformation scheme can be obtained from the diagram of figure 4 simply by reversing the arrows — limiting and deformation are inverse processes in a sense.

Actually, the Penrose diagram is not quite the same as ours since it contains further subdivisions of the Segre classification. Nevertheless, if one rejoins the subdivided cases the two diagrams become identical.

The limiting diagrams of the Petrov and the Segre classification play a fundamental role in the study of limits of space-times [17, 16, 15], as briefly discussed in

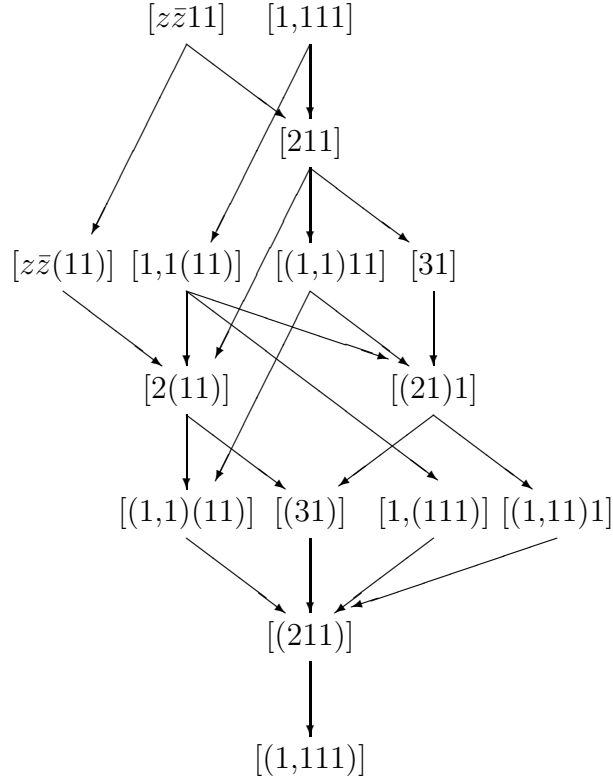


Figure 4: Diagram for the limits of the Segre types of the energy-momentum tensor in general relativity.

the introduction. The major result of the present work, i.e. the limiting diagram for the Segre types, also extends the coordinate-free approach to limits of vacuum space-times (where only the Petrov classification is relevant) to non-vacuum space-times, where the Segre classification plays an essential role.

The diagram in figure 3, based on the vanishing of scalars built with the metric and its derivatives together with the further subdivisions corresponding to the signs of certain scalars, show that the Segre classification meets Ehlers' requirement[5] that discrete invariants should be locally constant. This amounts to saying that the possible limits along curves in a given spacetime are the same as those for one-parameter families of spacetimes. Within a given spacetime,

if the Segre type at a generic point is known, more special types occur only on submanifolds of lower dimensions, as in the Petrov classification [23]. This justifies one of the steps in the spacetime classification scheme based on Cartan scalars [14].

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## Appendix A: Forbidden Limits of the Characteristic Polynomial

Our aim in this appendix is to discuss the three forbidden limits involving the characteristic polynomial with complex roots, which have been incorporated in the diagram of the left hand side in figure 2.

Let

$$D = (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_1 - \lambda_4)^2(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2(\lambda_3 - \lambda_4)^2 \quad (\text{A.1})$$

be the product of the squares of the differences of the 4 roots of the characteristic polynomial. By direct substitution of real, and then complex conjugate roots, one can easily show that  $D$  is positive for  $\{1111\}$  and negative for  $\{z\bar{z}11\}$ . Since  $D$  is built with the metric  $g$  and its derivatives, according to the hereditary property (3.1) the limit  $\{z\bar{z}11\} \rightarrow \{1111\}$  is forbidden.

Let now

$$M = (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 \quad (\text{A.2})$$

be the product of the squares of the differences of the 3 distinct roots of the characteristic polynomial  $\{211\}$  and  $\{2z\bar{z}\}$ . Similarly, by direct substitution of

real, and then of complex conjugate roots, one shows that  $M$  is positive for  $\{211\}$  and negative for  $\{2z\bar{z}\}$ . Again, using the hereditary property (3.1) the limit  $\{2z\bar{z}\} \rightarrow \{211\}$  is forbidden.

To deal with the limit  $\{2z\bar{z}\} \rightarrow \{31\}$  consider the following six scalars (built with the metric  $g$  and its derivatives):  $(\lambda_1 - \lambda_2)^2, (\lambda_1 - \lambda_3)^2, (\lambda_1 - \lambda_4)^2, (\lambda_2 - \lambda_3)^2, (\lambda_2 - \lambda_4)^2, (\lambda_3 - \lambda_4)^2$ , which are the squares of the differences of each pair of roots. For  $\{2z\bar{z}\}$  these scalars are such that one is a negative real number, another is zero, and the remaining four are two equal complex numbers and their complex conjugates. For  $\{31\}$  they are three equal real positive numbers, and three zeros. Now, from hereditary properties (2.1) and (3.1) one finds that if the limit  $\{2z\bar{z}\} \rightarrow \{31\}$  were permitted then the scalar which is zero would have to remain zero and the scalar which is negative would have to become zero, since  $\{31\}$  has no negative scalars. Further, to allow that limit, one out of the 4 complex scalars would have to become zero, and the remaining three would have to become equal positive real numbers, which clearly is not possible. Therefore the limit  $\{2z\bar{z}\} \rightarrow \{31\}$  is forbidden.

## Appendix B: Proofs of Hereditary Properties

Our aim in this appendix is to present proofs of the hereditary properties (3.2) and (3.3).

The property (3.2) can be proved as follows. Let  $S$  be the set of real space-time *directions* at a point  $p$  in the space-time manifold  $M$  and let  $N$  be the subset of real null directions at  $p$  with respect to the limit metric  $g$  at  $p$ . Let  $R$  be the limit Ricci tensor<sup>1</sup> at  $p$ . If  $k$  is a real vector at  $p$  and  $[k]$  the corresponding real

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<sup>1</sup>Clearly the Ricci tensor has the same Segre type as the trace-free Ricci tensor.



direction at  $p$ , so that  $[k] \in S$ , define a map  $f : S \mapsto \mathbb{R}$  by

$$f : [k] \mapsto \frac{P_{ab} P_{cd} h^{ac} h^{bd}}{(k^a k^b h_{ab})^2}, \quad (\text{B.3})$$

where  $h$  is a *positive definite* metric at  $p$  and

$$P_{ab} = k^c R_{c[a} k_{b]}. \quad (\text{B.4})$$

Then clearly  $f([k])$  is well defined on  $S$  and vanishes if and only if  $P_{ab} = 0$ , which is equivalent to  $k$  being an eigenvector of  $R$ . Now  $S (\approx P^3 \mathbb{R})$  is compact and  $N$ , being a closed subset of  $S$ , is also compact. So if  $R$  has no null eigenvectors at  $p$ ,  $f$  is nowhere zero on  $N$ . Since  $N$  is compact and connected and  $f$  continuous in the natural topologies on  $S$  and  $\mathbb{R}$  it follows that  $f$  is bounded away from zero. Hence there exists  $\varepsilon > 0$  such that  $f > \varepsilon$  on  $N$ . It follows that any “sequence” of Ricci tensors approaching  $R$  (and associated metrics approaching  $g$ ) must be such that they “eventually” have no real null eigenvectors. The result now follows by contradiction.

To prove the hereditary property (3.3) it is sufficient to show that the only types forbidden as a limit by (3.3) (i.e. the only remaining Segre types not admitting a null eigenvector) namely  $[1, (111)]$ ,  $[1, 1(11)]$  and  $[1, 111]$  cannot occur as a limit. Now the fact that  $[1, 111]$  cannot occur as such a limit can be seen either by considering minimal polynomials or by considering the quartic curves in the Penrose scheme [19]. The other types appear in the Ludwig-Scanlan classification scheme as

(a)	$[1, (111)]$	$A_2 \pm$
(b)	$[1, 1(11)]$	$B_{3f,g}$
(c)	$[1, 1(11)]$	$B_{5a,b}$
(d)	$[1, 1(11)]$	$C_1 \pm$

The possibility (b) can be eliminated by appealing again to the Penrose curve classification. To remove the remaining cases (a) (c) and (d) consider the continuous map  $f' : S \mapsto \mathbb{R}$  given in the above notation by

$$f' : [k] \longmapsto \frac{R_{ab} k^a k^b}{h_{ab} k^a k^b} . \quad (\text{B.5})$$

Then  $f'([k]) = 0 \iff R_{ab} k^a k^b = 0$  and so the points of  $N$  where  $f'$  vanishes are the real null vectors  $\ell$  satisfying  $R_{ab} \ell^a \ell^b = 0$  (the “generalized Debever-Penrose directions” [8], see also [3]). A similar argument to that given regarding the hereditary property (3.2) together with the facts that types (a), (c) and (d) above have no such generalized Debever-Penrose directions whereas the Segre types  $[z\bar{z}11]$  and  $[z\bar{z}(11)]$  have infinitely many [9] completes the proof.

## References

- [1] Bona C., Coll, B. & Morales J. A., “*Intrinsic Characterization of Space-time Symmetric Tensors*”, J. Math. Phys. **33**, 670 (1992).
- [2] Cajori, F. (1969), “*An Introduction to the Theory of Equations*”, Dover Publications, Inc, New York.
- [3] Cormack, W. J. & Hall, G. S. (1981), “*Spinors, Algebraic Geometry, and the Classification of Second-Order Symmetric Tensors in General Relativity*” Int. J. Theor. Phys. **20**, 105–119.
- [4] Crade, R. F. & Hall, G. S. (1982), “*Second Order Symmetric Tensors and Quadric Surfaces in General Relativity*” Acta Phys. Polonica **B13**, 405–419.
- [5] Ehlers, J. (1984), lecture given at the Georgenthal Relativity Seminar.  
Ehlers, J. (1981) “*Christoffel’s work on the equivalence problem for Rie-*

- mannian spaces and its importance for modern field theories of physics*", in *E.B. Christoffel*, ed. P.L. Butzer and F. Fehér, 526–542, Birkhäuser Verlag, Basel.
- [6] Geroch, R. (1969), "*Limits of Space-times*", Commun. Math. Phys. **13**, 180–193.
  - [7] Hall, G. S. (1982) "*Some Remarks on the Physical structure of Energy-Momentum Tensors in General Relativity*", Arch. Math. 4, Scripta Fac. Nat. Ujep Brunensis **XVIII**, 169 – 176.
  - [8] Hall, G. S. (1984), "*The Classification of Second Order Symmetric Tensors in General Relativity Theory*", Diff. Geom. **12**, 53–73.
  - [9] Hall, G. S. (1985), "*The Energy-Momentum Tensor in General Relativity*", Lecture notes, Balatonszéplak, Hungary.
  - [10] Hall, G. S. (1993), "*Physical and Geometrical Classification in General Relativity*", Brazilian Center for Physics Research Monograph (CBPF monograph) MO–001/93.
  - [11] Hall, G. S. & Negm, D. A. (1986), "*Physical Structure of the Energy-Momentum Tensor in General Relativity*", Int. J. Theor. Phys. **25**, 405–423.
  - [12] Hawking, S. W. & Ellis, G. F. R. (1973), "*The Large Scale Structure of Space-time*," Cambridge U. P., Cambridge.
  - [13] Ludwig, G. & Scanlan, G. (1971), "*Classification of the Ricci Tensor*", Commun. Math. Phys. **20**, 291–300.
  - [14] MacCallum, M.A.H. (1991), "*Computer-aided classification of exact solutions in general relativity*," in General relativity and gravitational physics

- (9th Italian Conference), ed. R. Cianci, R. de Ritis, M. Francaviglia, G. Marmo, C. Rubano and P. Scudellaro, pp. 318-337, World Scientific, Singapore.
- [15] Paiva, F. M. (1993), “*Limites de Espaços-tempos em Gravitação*”, Ph. D. thesis, Brazilian Center for Physics Research (CBPF), Rio de Janeiro.
  - [16] Paiva, F. M. & Romero, C. (1993), “*The Limits of Brans-Dicke Space-times: a Coordinate-free Approach*”, Gen. Rel. Grav. **25**, 1305–1317.
  - [17] Paiva, F. M., Rebouças, M. J. & MacCallum, M. A. H. (1993), “*On Limits of Space-times – a Coordinate-free Approach*”, Class. Quantum Grav. **10**, 1165–1178.
  - [18] Penrose, R. (1960), “*A Spinor Approach to General Relativity*”, Ann. Phys. NY **10**, 171–201.
  - [19] Penrose, R. (1972), “*Spinor classification of energy tensors*”, in Gravitation: Problems, Prospects (Dedicated to the memory of A. Z. Petrov). Izdat, Naukova Dumka, Kiev.
  - [20] Penrose, R. & Rindler, W. (1986), “*Spinors and Space-Time*”, vol. 2, Cambridge U. P., Cambridge.
  - [21] Petrov, A. Z. (1966), “*Perspectives in Geometry and Relativity*”, Essays in Honour of Václav Hlavatý, Indiana University Press, Bloomington and London.
  - [22] Petrov, A. Z. (1969), “*Einstein Spaces*”, first English edition, Pergamon Press.

- [23] Rendall, A. D. (1988), “*Curvature of Generic Space-times in General Relativity*”, J. Math. Phys. **29** 1569–1574.
- [24] Sánchez, A. R. G., Plebański, J. F. & Przanowski, M. (1991), “*Deformations of Algebraic Types of the Energy-momentum Tensor*”, J. Math. Phys. **32**, 2839–2847.
- [25] Santos, J., Rebouças, M. J. & Teixeira, A. F. F. (1995), “*Classification of Second Order Symmetric Tensors in 5-dimensional Kaluza-Klein-Type Theories*”, J. Math. Phys. **36**, 3074–3084.
- [26] Schmidt, H-J (1987), “*On Geroch’s limit of space-times and its relation to a new topology in the space of Lie groups*”, J. Math. Phys., **28**, 1928–1264.
- [27] Seixas, W. (1991), “*Extensions to the computer-aided Classification of the Ricci tensor*”, Class. Quantum Grav. **8**, 1577–1585.
- [28] Wilkinson, J. H. (1965), “*The Algebraic Eigenvalue Problem*”, Clarendon Press, Oxford.